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The problem of nonisothermal gas motion in a plane channel is considered for arbitrary values of the Knudsen number on the base of an S-model kinetic equation.

The rigorous theory of rarefied gas motion in channels for arbitrary values of the Knudsen number (Kn) can be constructed only on the basis of the solution of the Boltzmann kinetic equation or its models. Such solutions have been published in [1-6] in the case of a plane channel. The results obtained on the basis of the BGK model [1,2] need substantial refinement since this model does not describe the phenomena related to the simultaneous heat and mass transfer. An integral form of the kinetic equation with an ellipsoidal statistical model of the intermolecular collisions was used in [3,6]; results are obtained for iterations of the moment solution [3] and a variational Galerkin method [6]. The solution of the complete Boltzmann equation by the Monte Carlo method [4] differs substantially from all existing results, which is apparently due to the increase in the calculation error when taking account of intermolecular collisions. The Boltzmann equation is solved in [5] by the method of half-space moments in a third approximation. Since the moment method converges quite slowly for $Kn > 1$, the results obtained are reliable only for $Kn \leq 1$.

The problem of nonisothermal rarefied gas motion in a plane channel is solved in this paper on the basis of an S-model equation [7], which is a higher-order approximation compared to the BGK model. The purpose of the paper is, firstly, to obtain the correct solution of the problem in the whole range of values of the number Kn, secondly, to compare the results for different statistical models of the intermolecular collision operator, and thirdly, to estimate the effectiveness of the approximate methods of solving the transport integral equations to which the kinetic equation reduces.

Motion of a monatomic single-component gas is considered between infinite parallel planes $x = \pm d/2$, due to longitudinal pressure and temperature gradients. The state of the gas is assumed slightly perturbed, and therefore, described by a distribution function negligibly different from a Maxwell function. Fully diffuse scattering of the gas molecules by the channel walls is taken as boundary conditions. The S-model of the kinetic equation [7] is converted into a system of two linear integral equations for the heat flux density q and the macroscopic gas velocity U . This procedure is described in detail in [10], hence it is sufficient here to present just the final result which has the form

$$\psi = A\psi + B, \quad (1)$$

where

$$\psi = \begin{pmatrix} \psi_1 \\ \sqrt{\frac{\gamma}{15}} \psi_2 \end{pmatrix}; \quad B = \begin{pmatrix} \gamma \\ \sqrt{\frac{\gamma}{15}} \end{pmatrix};$$

$$A\psi = \frac{\delta}{\sqrt{\pi}} \int_{-1/2}^{+1/2} D\psi ds; \quad \delta = \frac{P_0}{\eta} d \left(\frac{m}{2kT_0} \right)^{1/2};$$

$$D = \begin{pmatrix} J_{-1} & \sqrt{\frac{2}{15}} \left(J_1 - \frac{1}{2} J_{-1} \right) \\ \sqrt{\frac{2}{15}} \left(J_1 - \frac{1}{2} J_{-1} \right) & \frac{2}{15} \left(J_3 - J_1 + \frac{9}{4} J_{-1} \right) \end{pmatrix}; \quad (2)$$

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$$\gamma = \frac{v}{\tau}; \quad v = \left(\frac{d}{P}\right) \frac{dP}{dz}; \quad \tau = \left(\frac{d}{T}\right) \frac{dT}{dz};$$

$$\psi_1 = -\gamma + \frac{2\delta}{\tau} \left(\frac{m}{2kT_0}\right)^{1/2} U; \quad \psi_2 = -1 + \frac{\delta}{\tau} \frac{4}{15} \left(\frac{m}{2kT_0}\right)^{1/2} \frac{q}{P_0};$$

$$J_n(t) = \int_0^{\infty} c^n \exp\left(-c^2 - \frac{t}{c}\right) dc,$$

the argument of the function J_n is $(\delta|x-s|)$; η is the coefficient of dynamic gas viscosity; P_0 , T_0 , gas pressure and temperature at the center of the channel for $z=0$; and k , Boltzmann's constant.

In order to establish the connection between the rarefaction parameter δ and the number Kn , it is necessary to write the expression for η . In particular, for the model of solid spherical molecules

$$\eta = \frac{1}{2} n_0 m \left(\frac{8kT_0}{\pi m}\right)^{1/2} l, \quad \delta = \frac{\sqrt{\pi}}{2} \frac{d}{l} = \frac{\sqrt{\pi}}{2} Kn^{-1}. \quad (3)$$

System (1) determines local values of the macroscopic gas velocity and the heat flux. However, the numerical I_m and thermal I_q fluxes averaged over the channel section are of practical interest:

$$I_m = n_0 \int_{-1/2}^{+1/2} U dx = L_{mm} X_m + L_{mq} X_q,$$

$$I_q = \int_{-1/2}^{+1/2} q dx = L_{qm} X_m + L_{qq} X_q, \quad (4)$$

where the thermodynamic forces have been selected in the form [1]

$$X_m = -kv, \quad X_q = -\frac{\tau}{T_0}. \quad (5)$$

The Onsager reciprocity relationship is valid for the averaged kinetic coefficients [1], i. e.,

$$L_{mq} = L_{qm}. \quad (6)$$

It is convenient to introduce dimensionless quantities for numerical computations, which are related to the kinetic coefficients by the following:

$$G_p = \left(\frac{m}{2kT_0}\right)^{1/2} \frac{2k}{n_0} L_{mm}, \quad G_\tau = -\left(\frac{m}{2kT_0}\right)^{1/2} \frac{2k}{P_0} L_{mq},$$

$$Q_p = -\frac{4}{15} \left(\frac{m}{2kT_0}\right)^{1/2} \frac{k}{P_0} L_{qm}, \quad Q_\tau = \frac{4}{15} \left(\frac{m}{2kT_0}\right)^{1/2} \frac{1}{P_0 T_0} L_{qq}, \quad (7)$$

where according to (6)

$$Q_p = -\frac{2}{15} G_\tau. \quad (8)$$

There exists a relationship between the fluxes I_m , I_q and the functions ψ_1 , ψ_2

$$I_m = \frac{n_0 \tau}{2\delta} \left(\frac{2kT_0}{m}\right)^{1/2} \left(\gamma + \int_{-1/2}^{+1/2} \psi_1 dx\right),$$

$$I_q = \frac{15}{4} \frac{\tau}{\delta} P_0 \left(\frac{2kT_0}{m}\right)^{1/2} \left(1 + \int_{-1/2}^{+1/2} \psi_2 dx\right). \quad (9)$$

An important particular case is the first-order stationary state when the temperature gradient is maintained constant and the appropriate pressure gradient is established in the system [1]. This phenomenon, called thermomolecular pressure, is characterized by the

TABLE 1. Numerical (G_p , G_T) and Thermal (Q_T) Fluxes for Two Statistical Models

δ	G_p			G_T			Q_T		
	S mod. variat.	S mod. numer.	BGK [1]	S mod. variat.	S mod. numer.	BGK [1]	S mod. variat.	S mod. numer.	BGK [1]
0,01	3,0517	3,0519	3,0489	1,2469	1,2470	1,2353	0,8979	0,8979	0,7230
0,1	2,0342	2,0397	2,0314	0,7283	0,7328	0,6744	0,5400	0,5407	0,4401
0,5	1,6144	1,6147	1,6017	0,4629	0,4630	0,4088	0,3189	0,3189	0,2521
1,0	1,5534	1,5541	1,5389	0,3653	0,3656	0,2953	0,2338	0,2339	0,1757
5,0	2,0029	2,0080	1,9883	0,1633	0,1642	0,1137	0,08217	0,08225	0,05013
10,0	2,7748	2,7863	2,7638	0,09707	0,09834	0,0655	0,04536	0,04543	0,02323

absence of a complete number flux averaged over the channel section [1-6], i.e., $I_m = 0$. Then there follows from (4), (5), (7), and (9)

$$\gamma = \frac{v}{\tau} = \left(\frac{T}{P} \frac{dP}{dT} \right) = \frac{G_T}{G_P} = - \int_{-1/2}^{+1/2} \psi_1 dx. \quad (10)$$

Therefore, the system of integral equations (1) must be solved to evaluate fluxes (9) and the index of thermomolecular pressure (10).

1. Krylov-Bogolyubov Numerical Method

This method [8] consists of using special quadrature formulas to reduce the integral equation to a system of algebraic equations.

Because of the symmetry of the problem, it is sufficient to consider the range of interactions (0, 1/2). Let x_0, x_1, \dots, x_n be points dividing this interval, where $x_0 = 0$ and $x_n = 1/2$. Then system (1) can be written in the form

$$\psi(x_{k+1/2}) = \frac{\delta}{\sqrt{\pi}} \sum_{j=0}^{n-1} \alpha_{kj} \psi(x_{j+1/2}) + B, \quad (11)$$

where

$$\alpha_{kj} = \int_{x_j}^{x_{j+1/2}} D(x_{k+1/2}, s) ds, \quad x_{k+1/2} = \frac{1}{2}(x_k + x_{k+1}), \quad k=0, 1, 2, \dots, n-1. \quad (12)$$

The quantity γ in B has the following form according to (10)

$$\gamma = -2 \sum_{k=0}^{n-1} \psi_1(x_{k+1/2}) \Delta x_k, \quad \Delta x_k = x_{k+1} - x_k. \quad (13)$$

For the best convergence of the method, the division points are selected so that the interval (x_k, x_{k+1}) would diminish with increasing k . This is due to the rapid change in the functions ψ_i near the channel surface.

2. Galerkin Method [9]

This method assumes the selection of a system of base functions. The symmetry of the problem requires that these functions be even, e.g., $\{x^{2k}\}$ ($k = 0, 1, \dots$). It is possible to limit oneself to the following approximation [6]

$$\tilde{\psi} = \left(\begin{array}{c} a + bx^2 \\ \sqrt{\frac{15}{2}} c \end{array} \right). \quad (14)$$

Such a form for the vector $\tilde{\psi}$ is valid in the approximation of a continuous medium and is a solution of the Navier-Stokes and heat-conduction equations.

Substitution of (14) in system (1) yields a residual of the form

$$R = \tilde{\psi} - A\tilde{\psi} - B. \quad (15)$$

The constants a, b, c are determined from the orthogonality conditions of the residuals to

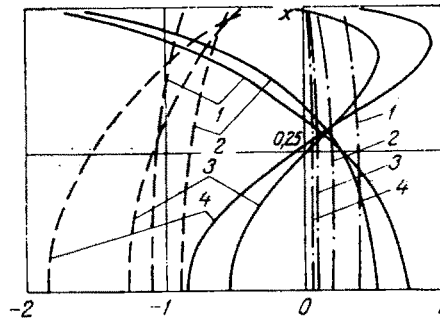


Fig. 1. Velocity profiles in channel sections: dashed curves) Poiseuille stream velocity profile; dash-dot curve) for thermal creep; solid curve) total velocity profile; 1-4) $\delta = 0.1$; 1.0; 5.0; 10.0.

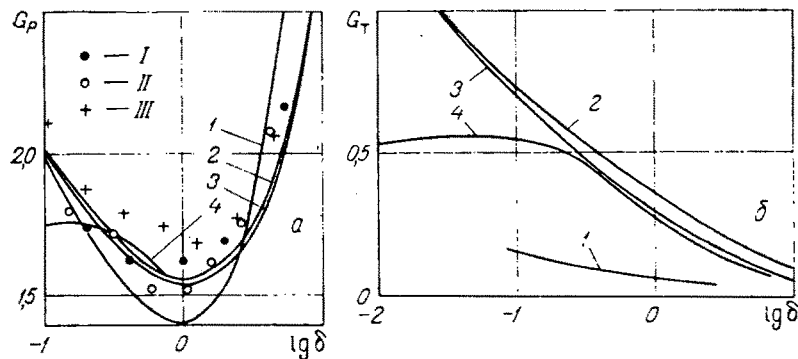


Fig. 2. Poiseuille flux (a) and thermal creep (b) as a function of the gas rarefaction parameter δ : 1) Maxwell sphere (Monte Carlo method [4]); 2) S-model; 3) BGK [1]; 4) Maxwell molecules (third approximation of the half-space moment model [5]); I) H_2 ; II) Ar; III) He [13].

the selected base functions, i.e.,

$$\int_{-1/2}^{+1/2} \mathbf{E}_1 \mathbf{R} dx = 0, \quad \int_{-1/2}^{+1/2} \mathbf{E}_2 \mathbf{R} dx = 0, \quad \int_{-1/2}^{+1/2} \mathbf{E}_3 \mathbf{R} dx = 0, \quad (16)$$

where

$$\mathbf{E}_1 = (1 \ 0); \quad \mathbf{E}_2 = (x^2 \ 0); \quad \mathbf{E}_3 = (0 \ 1).$$

The magnitudes of the fluxes I_m , I_q and the thermomolecular pressure index γ are evaluated after (14) has been substituted into (9) and (10).

Numerical computations were performed on a BESM-6 computer. Confirmation of the convergence of the Krylov-Bogolyubov method showed that the error in calculating the macroscopic quantities for $n = 24$ is less than 0.1% for any values of δ . Hence, the numerical solution can be taken as a standard in comparison with other approximate solutions.

Results for dimensionless Poiseuille flow G_p , thermal creep G_T , and the heat flux by heat conduction Q_T obtained by the numerical and variational methods, as well as the data from [1] for the BGK model are presented in Table 1. It is seen that the maximum relative deflection of the variational from the standard solution for $\delta = 10$ does not exceed 1.5%. The main disadvantage of the Galerkin method is that it is difficult to use to calculate the local values of the macroparameters, while its advantage is a smaller expenditure of machine time.

It follows from a comparison of results for different models that the divergence in G_p

reaches 1% for $\delta \approx 2$, 31% in G_T at $\delta = 10$, and 49% in Q_T for $\delta = 10$. This discrepancy is explained by the fact that the BGK model does not permit description of the processes due to simultaneous gas viscosity and heat conduction. The results may be refined if problems of Poiseuille flow and thermal creep are solved in isolation by selecting different collision parameters in the BGK-model [12]. However, even in this case the discrepancy in the heat flux Q_T reaches 15% for the different models.

The velocity profiles of Poiseuille flow u_p/v , the thermal creep u_T/τ , and also the total velocity profile

$$\frac{u}{\tau} = \gamma \frac{u_p}{v} + \frac{u_T}{\tau} \quad (17)$$

are presented in Fig. 1, hence all the quantities are referred to the logarithmic pressure v and temperature τ gradients, and are measured in the units $(m/2kT_0)^{1/2}$. It is interesting to note that the total gas flux at both the center and at the walls of the channel has a direction dependent on the rarefaction parameter δ . As δ increases the contribution of the Poiseuille flux increases at the center of the channel and of thermal creep at the circumference.

Different theories for Poiseuille flow (Fig. 2a) and thermal creep (Fig. 2b) are compared in Fig. 2. Reasons for the occurrence of the results are discussed at the beginning of the paper.

Comparing theory with experiment for Poiseuille flow [13] (Fig. 2a) is of interest. Satisfactory agreement is observed for $\delta > 1$. The discrepancy both between the experimental data for different gases and between theory and experiment for $\delta < 1$ is due firstly to the difference in the accommodations of the gas molecules at the channel walls, and secondly to the finite dimensions of the experimental channel.

NOTATION

Kn , Knudsen number; d , spacing between plates; P , pressure; T , temperature; v , logarithmic pressure gradient; τ , logarithmic temperature gradient; U , macroscopic gas velocity; q , heat flux; η , coefficient of dynamic viscosity; \bar{l} , mean free path; I_m , numerical gas flux averaged over the channel section; I_q , gas heat flux averaged over the channel section; γ , universal index of the thermomolecular pressure difference; u_p , Poiseuille flux velocity; and u_T , thermal creep rate.

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